1990 Calculus AB

Q1 (a) $a(t) = 12t^2 - 4$

$v(t) = \int 12t^2 - 4\,dt$

$v(t) = 4t^3 - 4t + c$

$v(0) = 0 \Rightarrow c = 0$

$\boxed{v(t) = 4t^3 - 4t}$

PARTICLE IS AT REST WHEN $v(t) = 0$

$\Rightarrow 4t(t^2 - 1) = 0$

$t = 0, \quad t = \pm 1$

$\boxed{\text{Answer } 0, 1}$

(b) $x(t) = \int 4t^3 - 4t\,dt$

$x(t) = t^4 - 2t^2 + c$

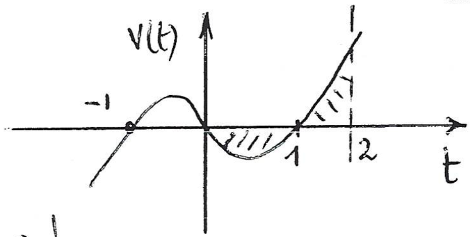
$x(1) = 3$

$\Rightarrow 3 = 1 - 2 + c$

$c = 4$

$\Rightarrow \boxed{x(t) = t^4 - 2t^2 + 4}$ — Answer

(c) Either



$D = \text{Total Distance} = \int_0^1 0 - (4t^3 - 4t)\,dt + \int_1^2 4t^3 - 4t\,dt$

$D = \left(-t^4 + 2t^2\right)_0^1 + \left(t^4 - 2t^2\right)_1^2$

$\boxed{D = 10}$

$\boxed{\text{OR}}$

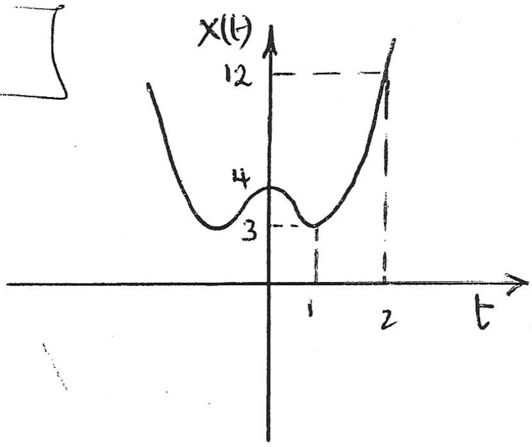
$x(t) = t^4 - 2t^2 + 4$

Total Distance $= x(1) - x(0) + x(2) - x(1)$

$D = |3 - 4| + 12 - 3$

$\boxed{D = 10}$

$\boxed{\text{NB}}$ THE PARTICLE MOVES TO THE LEFT FOR $0 < t < 1$ AND THEN



Q2 (a) $f(x) = \ln\left(\dfrac{x}{x-1}\right)$

$\dfrac{x}{x-1} > 0 \Rightarrow$ + (0) − (1) +

Domain: $(-\infty, 0) \cup (1, \infty)$

(b) $$f'(x) = \frac{1}{\frac{x}{x-1}} \cdot \left\{ \frac{(x-1)(1) - x(1)}{(x-1)^2} \right\}$$

$$= \left(\frac{\cancel{x-1}}{x} \right) \left\{ \frac{x-1-x}{(x-1)^2} \right\}$$

$$= \frac{-1}{x(x-1)}$$

$$f(-1) = \frac{-1}{(-1)(-2)} = -\frac{1}{2}$$

(c)

Let $y = \ln\left(\frac{x}{x-1}\right)$

$$\frac{x}{x-1} = e^y$$

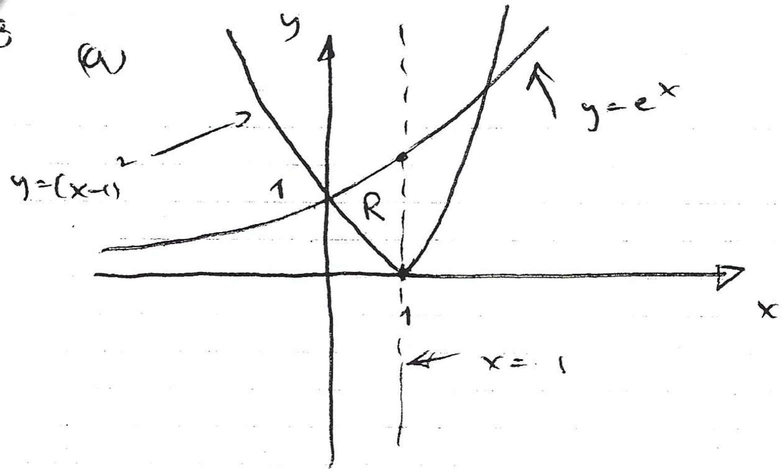
$$x = xe^y - e^y \qquad x \neq 1$$

$$x(1-e^y) = -e^y$$

$$x = \frac{-e^y}{1-e^y}$$

$$\boxed{f^{-1} : x \mapsto \frac{-e^x}{1-e^x}}$$

Q3 (a)



(a)

$$A=\int_0^1 e^x\,dx-\int_0^1 (x-1)^2\,dx$$

$$=\left(e^x\right)_0^1-\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$=e-1-\left\{0+\tfrac{1}{3}\right\}$$

$$=e-\tfrac{4}{3}$$

Answer:

(b) $V=\pi\int_0^1 (e^x)^2-(x-1)^4\,dx$

$$V=\pi\left[\frac{e^{2x}}{2}-\frac{(x-1)^5}{5}\right]_0^1$$

$$V=\pi\left[\frac{e^2}{2}-0\right]-\pi\left[\frac{1}{2}+\frac{1}{5}\right]$$

$$=\frac{\pi e^2}{2}-\frac{7\pi}{10}$$

$$=\pi\left(\frac{5e^2-1}{10}\right)$$

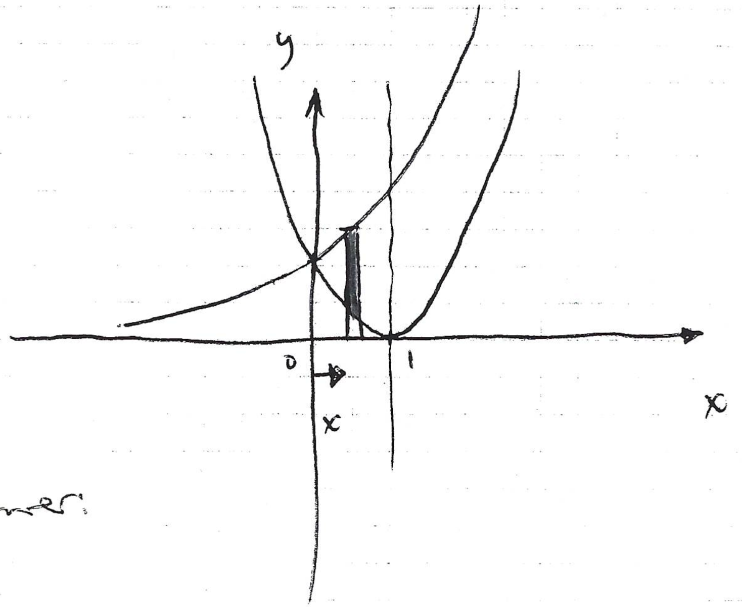
(c)

Shell method:

$$V=2\pi\int_0^1 x\left[e^x-(x-1)^2\right]dx$$

$$V=2\pi\int_0^1 xe^x-x(x-1)^2\,dx$$

Answer:



Disc:

$$V=\pi\int_0^1 1-(1-\sqrt{y})^2\,dy+\pi\int_1^e 1-(\ln y)^2\,dy$$

Q4 $\frac{dr}{dt} = 0.04$ cm/s

(a) when $r=10$

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

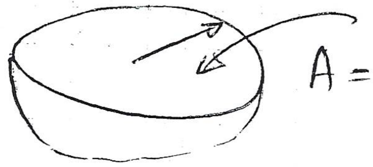
$$= (4\pi r^2)\frac{dr}{dt}$$

when $r=10$,

$$\Rightarrow \frac{dV}{dt} = 4\pi(10)^2(0.04)$$

$$\boxed{\frac{dV}{dt} = 16\pi} \text{ cm}^3/\text{s}$$

(b)



$A = \pi r^2$

when $V = 36\pi$

$$\Rightarrow 36\cancel{\pi} = \frac{4}{3}\cancel{\pi} r^3$$

$$27 = r^3$$

$$\boxed{r = 3}$$

$$\Rightarrow \frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$= 2\pi r \cdot \frac{dr}{dt}$$

when $r=3$, $\frac{dr}{dt} = 0.04$

$$\Rightarrow \frac{dA}{dt} = 2\pi(3) \cdot (0.04)$$

$$= 0.24\pi$$

$$\boxed{\frac{dA}{dt} = 0.24\pi \text{ cm}^2/\text{s}}$$

(c) when $\frac{dV}{dt} = \frac{dr}{dt} = 0.04$

$$\Rightarrow 4\pi r^2 \frac{dr}{dt} = \frac{dV}{dt} = 0.04$$

$$\Rightarrow 4\pi r^2 = 1$$

$$r^2 = \frac{1}{4\pi}$$

$$r = \pm\sqrt{\frac{1}{4\pi}}$$

$r>0$, $r = \frac{1}{2\sqrt{\pi}} = \frac{\sqrt{\pi}}{2\pi}$ cm.

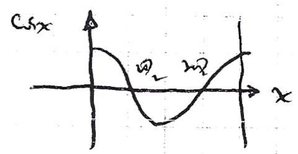
15 $\quad f(x) = \sin^2 x - \sin x \qquad 0 \leq x \leq \frac{3\pi}{2}$

(a) $\quad f(x) = 0, \quad \sin x(\sin x - 1) = 0$

$\sin x = 0 \qquad \sin x = 1$

$\boxed{x = 0, \pi} \qquad \boxed{x = \frac{\pi}{2}}$

(b) $\quad f'(x) = 2\sin x \cos x - \cos x$

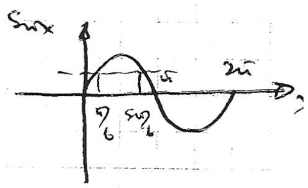


$f'(x) > 0, \quad (2\sin x - 1)\cdot\cos x > 0$

either

$\Rightarrow \quad 2\sin x - 1 > 0$ And $\cos x > 0$ OR $2\sin x - 1 < 0$ And $\cos x < 0$

$\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ And $0 \leq x \leq \frac{\pi}{2}$



$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

gives $\quad \frac{\pi}{6} \leq x \leq \frac{\pi}{2}$ $\qquad$ OR $\quad x \leq \frac{\pi}{6}$ OR $x \geq \frac{5\pi}{6}$ And $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

gives $\quad \frac{5\pi}{6} \leq x \leq \frac{3\pi}{2}$

Answer $\quad \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ OR $\left[\frac{5\pi}{6}, \frac{3\pi}{2}\right]$

(c) $\quad f'(x) = 0, \quad (2\sin x - 1)\cos x = 0$

$\sin x = \frac{1}{2} \qquad \cos x = 0$

$x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad x = \frac{\pi}{2}, \frac{3\pi}{2}$

$f'(x) = \sin 2x - \cos x$

$f''(x) = 2\cos 2x + \sin x$

$f''(\pi/2) = -2 + 1 = -1$ [MAX] $\Rightarrow f(\pi/2) = 0$

NOT MAX

~~$f''(3\pi/2) = -2 - 1 = -3$ [MAX] $\Rightarrow f(3\pi/2) = 2$~~

$f''(\pi/6) = 2\cos \pi/3 + \sin \pi/6 > 0$ [MIN] $\Rightarrow f(\pi/6) = -1/4$

$f''(5\pi/6) = 2\cos(5\pi/3) + \sin 5\pi/6 > 0$ [MIN] $\Rightarrow f(5\pi/6) = -1/2$

without [illegible]

$f(0) = 0$

$f(3\pi/2) = 2$

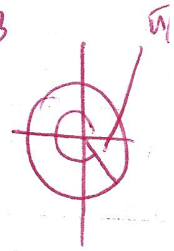
Absol. Max is 2

Absol. Min is $-1/2$.

$\cos 5\pi/3$

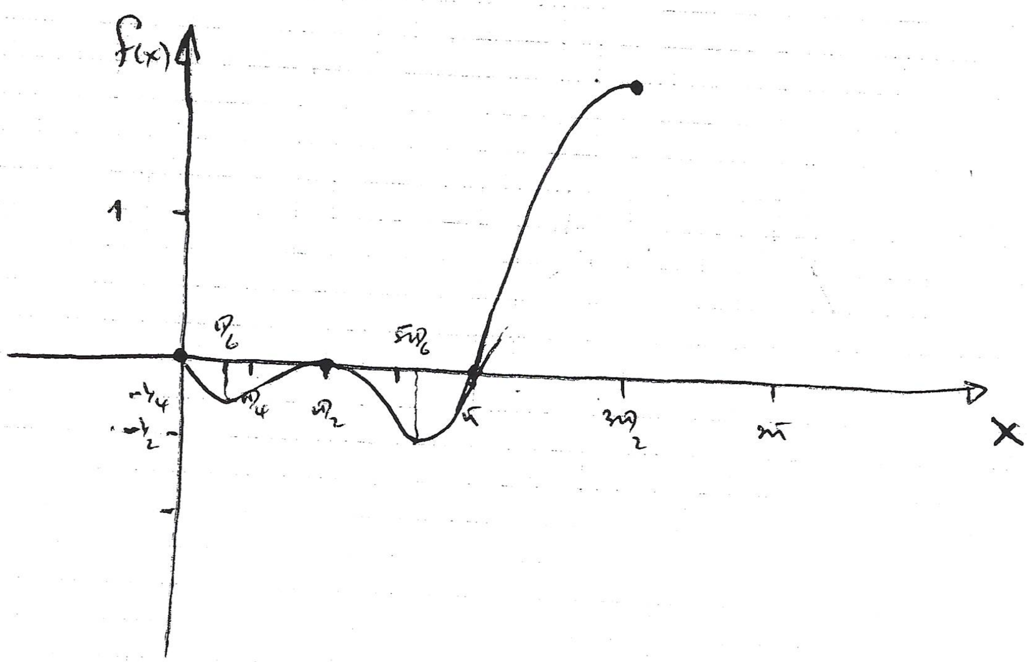
$= \cos \pi/3$

$= 1/2$

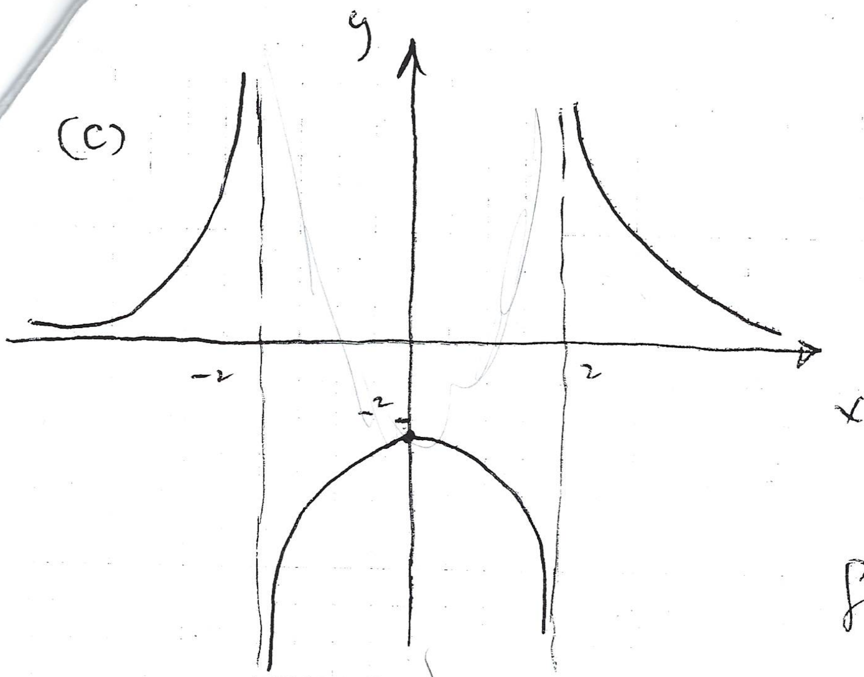


Extra:

Sketch:



(c)



$$f(x) = \frac{9}{x^2-4}$$

how do you know it goes down for not up

NB $f'(x) = \dfrac{-18x}{(x^2-4)^2}$ $f'(1) = -2$

Turning pt. @ $(0, -9/4)$

when $x = -1/2$, $f'(-1/2) > 0$

$x = 1/2$, $f'(1/2) < 0$

$\Rightarrow$ Rel max value @ $x = 0$.

$$\lim_{x \to -2^+} \left(\frac{9}{x^2-4}\right) = -\infty$$

Q6 $f(x) = \dfrac{ax+b}{x^2-c}$

(a) (i) $f(x)$ is Even $\Rightarrow$ $\boxed{a=0}$ Power of x must be Even.

(ii) $\lim\limits_{x \to 2^+} f(x) = +\infty$

$\Rightarrow \boxed{c=4}$

(iii) $f'(x) = \dfrac{(x^2-c)(a)-(ax+b)(2x)}{(x^2-c)^2}$

$a=0$, $c=4$, $f'(1) = \dfrac{-2b}{(1-4)^2} = -2$

$-2b = -18$

$\boxed{b=9}$

(b) $f(x) = \dfrac{9}{x^2-4}$

Vert Asymptotes:

$x^2-4=0$

$\boxed{x=2}$ $\boxed{x=-2}$

$\boxed{\text{Horiz}}$

$\boxed{y=0}$